About the same time, the subject of scattering of elastic waves in solids was revived in the fields of acoustics and geophysics for the purpose of studying energy losses as sound waves pass through obstacles in an elastic matrix. In 1956, Ying and Truell investigated the scattering of plane waves by a spherical obstacle,(1.79) and later, White (1958) studied the scattering at a cylindrical discontinuity with experimental observations.(1.80) Both were dealing with waves varying harmonically in time. The problem of a spherical obstacle has also been treated by Takeuchi(1.81) and by Knopoff.(1.82) If the incident wave were a pulse, the analysis becomes more complicated. In 1959, Gilbert and Knopoff provided an asymptotic solution (early time) for the diffraction of a step pulse by a circular cylinder.(1.83)

Once the true cause of stress concentration was identified with the diffraction of elastic waves, this knowledge was immediately applied to new findings. Interestingly, experimental investigation of dynamic stress concentration was initiated independent of theoretical analysis at about the same time. Using the method of dynamic photoelasticity, Wells and Post (1958) investigated the dynamic stress distribution surrounding a running crack,(1.84) and Durelli and Dally (1959) measured the dynamic stress concentration factor at a central circular hole in a plate subjected to axial impact.(1.85) Research activity has greatly accelerated in the sixties, resulting inevitably in a flourish of publications, most of which will be reviewed in the subsequent chapters.

2. ELEMENTS OF THE THEORY OF ELASTICITY

AS MENTIONED IN THE HISTORICAL INTRODUCTION, the theory of elasticity had been well developed by the middle of the 19th century. About that time, several treatises on the subject appeared in different languages. In English, the first edition of *Mathematical Theory of Elasticity* by A. E. H. Love was published in 1892.* The book was revised three times and the last edition appeared in 1927. It has since been reprinted many times and is still used widely. Many textbooks on the same subject are also available, of which we mention the one by Timoshenko and Goodier (1969),(1.61) by Sokolnikoff (1956),(2.21) and by Green and Zerna (1968).(2.2) All the books cited above contain excellent accounts of the general theory of elasticity: there is no need to describe it here. In this section and the next, only the parts of the general theory, and equations which are basic to investigating the scattering of elastic waves, will be presented, and only briefly.

2.1. *Deformation, Displacements, and Strains*

A continuous body subjected to the action of external forces occupies different regions in space from time to time. Let the regions be referred to a Cartesian coordinates system, fixed in space. Every particle of the body is identified initially by the coordinates $a_i (i = 1, 2, 3)$. If the particle is carried to a new position with coordinate $x_i (i = 1, 2, 3)$, the transformation

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*Thomson (Lord Kelvin) and Tait's *Treatise on Natural Philosophy* appeared earlier (in 1879) but it was much broader in scope and covered rigid body statics and dynamics, elasticity, and hydrodynamics. W. J. Ibbetson's *Mathematical Theory of Perfectly Elastic Solids* (in 1887) was much less in contents.

$$x_i = x_i(a_1, a_2, a_3, t), \qquad i = 1,2,3, \tag{2.1a}$$

where t is considered as a parameter, is called the *deformation*. We assume that the inverse transformation

$$a_i = a_i(x_1, x_2, x_3, t), \qquad i = 1,2,3 \tag{2.1b}$$

uniquely exists. The directed line segment which connects these two positions of the same particle is called the displacement, a vector quantity with components u_i. As can be seen from Fig. 2.1,

$$u_i = x_i - a_i. \tag{2.2}$$

By applying Eq. (2.1), u_i can be expressed as a function either of the material coordinates a_i, or of the spatial coordinates x_i.

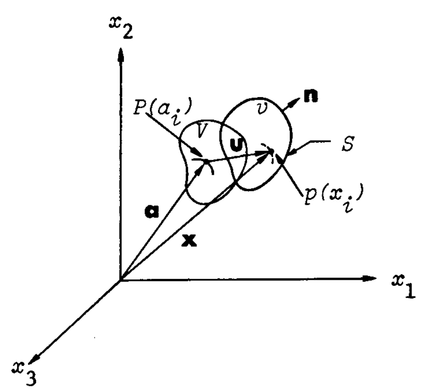

Fig. 2.1. Geometry of Deformation

The change in length and in relative direction between two adjacent particles accompanying the deformation is called *strain*. To study strains, it is necessary to consider the local behavior of the deformation. We define

$$dx_i = \frac{\partial x_i}{\partial a_j}\, da_j; \qquad da_i = \frac{\partial a_i}{\partial x_j}\, dx_j. \tag{2.3}$$

(The summation convention over repeated indices is adopted throughout.) By using Eq. (2.2), the coefficients of differentials in (2.3), known as *deformation gradients*, can be expressed as

$$\frac{\partial x_i}{\partial a_j} = \delta_{ij} + \frac{\partial u_i}{\partial a_j}\,; \qquad \frac{\partial a_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j}\,, \tag{2.4}$$

where δ_{ij} is the Kronecker delta which has the value one when $i = j$, and zero when $i \neq j$. The quantities $\partial u_i/\partial a_j$ and $\partial u_i/\partial x_j$ are the *displacement gradients* referred to the material coordinates and spatial coordinates respectively. We note that the part δ_{ij} represents a *translation* which shifts dx_i to da_j or vice versa.

A body is *rigid* if the distance between any two particles remains unchanged, i.e.,

$$dx_i\, dx_i = da_i\, da_i.$$

The displacement for a rigid body is composed of a translation of a particle of the body plus a rotation of the body about that particle.

The body is said to be strained if the relative positions of any two particles of the body change. A strained body is *elastic* if, after the external agents that induce the strain have been removed, the body recovers its original shape and size. In other words, the region v occupied by the deformed body (Fig. 2.1) can be brought to coincide with the region V through a rigid-body displacement after the removal of external forces. Elasticity theory is concerned with the deformation of an elastic body.

In the linear theory of elasticity, it is assumed that the displacement gradients are small, so that the product

$$\frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j} \quad \text{and} \quad \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}$$

can be neglected in comparison with $\partial u_i/\partial a_j$ and $\partial u_i/\partial x_j$ respectively. Furthermore, the product

$$\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial a_j}$$

is also neglected. Thus

$$\frac{\partial u_i}{\partial a_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial a_j} = \frac{\partial u_i}{\partial x_k}\left(\delta_{kj} + \frac{\partial u_k}{\partial a_j}\right) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial a_j} \simeq \frac{\partial u_i}{\partial x_j},$$

which implies that the material and spatial displacement gradients approximate each other. This enables us to establish the kinetic equations for a strained elastic body with reference to either the deformed or undeformed region. Henceforth, we shall use spatial description.

Equations (2.3) and (2.4) show that the local behavior of deformation is characterized by a translation and the displacement gradient $\partial u_i/\partial x_j$. We decompose the latter into the symmetric and antisymmetric parts:

$$\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + \omega_{ij},$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.5}$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{2.6}$$

The symmetric part ε_{ij} is the *small strain tensor*. It characterizes the change in length and relative position of line elements. The antisymmetric part ω_{ij} is the *small rotation tensor*. It represents the average rotation of two line elements initially at right angles to each other. There are only three nonvanishing components of ω_{ij}, which can also be derived from the curl of the vector $\mathbf{u}$ by $\boldsymbol{\omega} = -\frac{1}{2}\nabla \times \mathbf{u}$, or from

$$\omega_i = -\tfrac{1}{2}e_{ijk}\frac{\partial u_k}{\partial x_j}. \tag{2.7}$$

(e_{ijk} is the permutation symbol which has the value +1 when the indices are in cyclic order (123, 231, 312); -1 when in acyclic order (321, 213, 132); and zero otherwise.) The components of the antisymmetric tensor ω_{ij} in (2.6) and those of an axial vector ω_i in (2.7) are connected by the relation

$$\omega_{12} = \omega_3, \qquad \omega_{23} = \omega_1, \qquad \omega_{31} = \omega_2.$$

The three components ε_{11}, ε_{22}, ε_{33} of the small strain tensor are the *normal strains*, whereas the remaining elements ε_{12}, ε_{13}, ε_{23} are the *shear strains*. The sum of the three normal strains, which is also the first scalar invariant of the strain tensor, is called *cubical dilatation* or simply *dilatation*,

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \partial u_k/\partial x_k. \qquad (2.8)$$

It equals approximately the ratio of the increment of volume to the original volume of a cubical element.

To assure the single-valueness of displacement upon the integration of Eq. (2.5), the six strain components must satisfy the *compatibility conditions*

$$\varepsilon_{ij,k\ell} + \varepsilon_{k\ell,ij} - \varepsilon_{ik,j\ell} - \varepsilon_{j\ell,ik} = 0. \qquad (2.9)$$

In problems of elastodynamics, solutions are usually found directly in terms of displacements (see subsection 2.4). Thus, if the displacements are continuous and single-valued functions, the strains derived from Eq. (2.5) are always compatible.

2.2. *Forces, Tractions, and Stresses*

Deformation of an elastic body has among its causes the action of a *force*, a vector quantity. Forces that act throughout the body and are generally proportional to the masses of the particles are called *body forces*. Those that act over a surface of the body are *surface forces*.

If a deformed body is divided into two parts by a fictitious plane, the action of one part of the body on the other can be represented by a force and a couple. Since the action varies from point to point on the plane, we introduce the *surface traction* and the *surface couple* to represent the action at a point on the plane. The integral of the surface traction over the entire plane equals the total force. Surface traction, a vector, will be denoted by $\mathbf{T}^{(n)}$, where $\mathbf{n}$

is a unit vector normal to the plane. The total couple on the plane then equals an integration of the *surface couple* over the entire area. In the classical theory of elasticity, the surface couple is assumed to be of negligible importance. When two bodies are mutually in contact, the action of one body on the other through the contact surface can also be represented by surface traction $\mathbf{T}^{(n)}$.

The traction not only changes from point to point, it also depends on the orientation of the plane through a given point. It can be shown that at a point, if the tractions acting on three mutually perpendicular planes ($\mathbf{T}^{(1)}$, $\mathbf{T}^{(2)}$, and $\mathbf{T}^{(3)}$) are known, the traction over any other plane with direction normal $\mathbf{n}$ can be calculated by

$$\mathbf{T}^{(n)} = n_1\mathbf{T}^{(1)} + n_2\mathbf{T}^{(2)} + n_3\mathbf{T}^{(3)}.$$

The n_i are the components of $\mathbf{n}$ along the coordinate axes (x_1, x_2, x_3) formed by the three planes. Since each vector may be resolved into components along the three axes, the components of the traction on a plane with normal $\mathbf{n}$ are given by

$$T_j^{(n)} = n_1T_j^{(1)} + n_2T_j^{(2)} + n_3T_j^{(3)} = n_iT_j^{(i)}, \qquad i,j = 1,2,3.$$

When expressed in terms of a new symbol σ_{ij}

$$T_j^{(i)} \equiv \sigma_{ij}, \qquad i,j = 1,2,3, \qquad (2.10)$$

we have

$$T_j^{(n)} = n_i\sigma_{ij}. \qquad (2.11)$$

The nine components σ_{ij} of tractions at three surfaces form a *stress* *tensor* which specifies completely the force action at a point in a deformed body.

The components σ_{11}, σ_{22}, σ_{33} are normal stresses. Each is the normal component of the traction acting on the surface perpendicular to x_i-axis. The remaining six tangential components are *shear stresses*.

2.3. *Equations of Motion and Constitutive Equations*

The equations of motion for an elastic body are derived from the principles of balance of momentum, balance of angular momentum, and balance of energy. Consider an infinitesimal element of an elastic body with mass density ρ being subjected to a body force per unit mass **f**. From the balance of momentum is deduced the equation of motion

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j = \rho \frac{\partial^2 u_j}{\partial t^2}. \tag{2.12}$$

The balance of angular momentum leads to the result

$$e_{ijk}\sigma_{jk} = 0,$$

which implies that

$$\sigma_{jk} = \sigma_{kj} \tag{2.13}$$

or that the stress tensor is symmetric.

The principle of balance of energy states that the rate of change of energy equals the rate of work done by the external forces acting in the elastic body. Let W be the internal energy per unit of volume of a strained body. By applying this principle it can be shown that

$$\frac{dW}{dt} = \sigma_{ij}\frac{d\varepsilon_{ij}}{dt}. \tag{2.14}$$

If in addition, the internal energy density W is a function of nine strain components,

$$W = W(\varepsilon_{ij}), \tag{2.15}$$

it then follows from the equation of balance of energy (2.14) that

$$\sigma_{ij} = \frac{\partial W(\varepsilon_{ij})}{\partial \varepsilon_{ij}}.$$

Since both strain and stress tensors are symmetric, we write

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial \varepsilon_{ji}}\right). \tag{2.16}$$

Equation (2.15) defines the property of the material and it is known as the *constitutive equation* for an elastic medium. The linear theory further assumes that W is a quadratic function of ε_{ij}

$$W = \tfrac{1}{2}\, c_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell}, \tag{2.17}$$

where $c_{ijk\ell}$ are constant coefficients if the material is homogeneous. Substitution of (2.17) in (2.16) results in a stress-strain relation

$$\sigma_{ij} = c_{ijk\ell}\varepsilon_{k\ell}, \tag{2.18}$$

which is known as the *generalized Hooke's law* for an aelotropic, homogeneous elastic material. Because of the assumed quadratic form (2.17) and the symmetry of σ_{ij} and ε_{ij}, the elastic constants $c_{ijk\ell}$ possess the following symmetry

$$c_{ijk\ell} = c_{k\ell ij} = c_{jik\ell} = c_{ij\ell k}. \tag{2.19}$$

Thus out of 81 components of $c_{ijk\ell}$, only 21 are independent.

If the material is also isotropic

$$c_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}),$$

where λ and μ are known as Lamé's constants and the sress-strain relation assumes the simple form

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}. \tag{2.20}$$

Instead of λ and μ, other constants are also used to represent the elastic property of an isotropic, homogeneous material. The common ones are Poisson's ratio ν, Young's modulus E, and bulk modulus k. Any one of these constants can be expressed in terms of the other two as shown in Table 2.1. Since μ relates the shearing stresses $\sigma_{ij}(i \neq j)$ to shearing strains, it is also called shear modulus.

2.4. *Boundary Value Problems of Elasticity*

In the previous subsections, we have shown that the components of stress tensor σ_{ij}, strain tensor ε_{ij}, and displacement vector u_j, a total of 15 unknown quantities, are related by the following 15 equations:

$$\frac{\partial\sigma_{ij}}{\partial x_i} + \rho f_j = \rho\,\frac{\partial^2 u_j}{\partial t^2}, \tag{2.12}$$

$$\sigma_{ij} = c_{ijk\ell}\varepsilon_{k\ell}, \tag{2.18}$$

Table 2.1

EQUIVALENCE OF ELASTIC CONSTANTS

Constants	λ	μ	E	ν	k
λ,μ			$\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$	$\frac{\lambda}{2(\lambda + \mu)}$	$\lambda + \frac{2}{3}\mu$
λ,ν		$\frac{\lambda(1 - 2\nu)}{2\nu}$	$\frac{\lambda(1 + \nu)(1 - 2\nu)}{\nu}$		$\frac{\lambda(1 + \nu)}{3\nu}$
λ,k		$\frac{3}{2}(k - \lambda)$	$\frac{9k(k - \lambda)}{3k - \lambda}$	$\frac{\lambda}{3k - \lambda}$	
μ,E	$\frac{\mu(E - 2\mu)}{3\mu - E}$			$\frac{E}{2\mu} - 1$	$\frac{\mu E}{3(3\mu - E)}$
μ,ν	$\frac{2\mu\nu}{1 - 2\nu}$		$2\mu(1 + \nu)$		$\frac{2\mu(1 + \nu)}{3(1 - 2\nu)}$
μ,k	$k - \frac{2}{3}\mu$		$\frac{9k\mu}{3k + \mu}$	$\frac{3k - 2\mu}{2(3k + \mu)}$	
E,ν	$\frac{E\nu}{(1 + \nu)(1 - 2\nu)}$	$\frac{E}{2(1 + \nu)}$			$\frac{E}{3(1 - 2\nu)}$
E,k	$\frac{3k(3k - E)}{9k - E}$	$\frac{3kE}{9k - E}$		$\frac{3k - E}{6k}$	
ν,k	$\frac{3k\nu}{1 + \nu}$	$\frac{3k(1 - 2\nu)}{2(1 + \nu)}$	$3k(1 - 2\nu)$		

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.5}$$

The body force f_j, density ρ, and elastic constants $c_{ijk\ell}$ are assumed to be given. Elimination of ε_{ij} and then σ_{ij} from the above equations leads to

$$\sigma_{ij} = c_{ijk\ell}\,\frac{\partial u_k}{\partial x_\ell}, \tag{2.21}$$

and finally,

$$c_{ijk\ell}\,\frac{\partial^2 u_k}{\partial x_i \partial x_\ell} + \rho f_j = \rho\,\frac{\partial^2 u_j}{\partial t^2}. \tag{2.22}$$

This is the equation of motion in terms of the displacement vector for a homogeneous elastic body.

For an isotropic material, $c_{ijk\ell}$ is replaced by two constants, λ, μ, as in (2.20). From the displacement components, stresses can be computed directly by

$$\sigma_{ij} = \lambda\,\frac{\partial u_k}{\partial x_k}\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i,j,k = 1,2,3, \tag{2.23}$$

and the equation of motion is

$$(\lambda + \mu)\,\frac{\partial^2 u_i}{\partial x_i \partial x_j} + \mu\,\frac{\partial^2 u_j}{\partial x_i \partial x_i} + \rho f_j = \rho\,\frac{\partial^2 u_j}{\partial t^2}, \qquad i,j = 1,2,3. \tag{2.24}$$

The above equations are expressed in terms of Cartesian components of vectors $\mathbf{u}$ and $\mathbf{f}$, and the stress tensor $\underline{\underline{\sigma}}$. They can also be expressed in terms of the vectors and the tensor themselves as

$$\underline{\underline{\sigma}} = \lambda(\nabla \cdot \mathbf{u})\underline{\underline{I}} + \mu(\nabla\mathbf{u} + \mathbf{u}\nabla), \tag{2.25}$$

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} + \rho\mathbf{f} = \rho\ddot{\mathbf{u}}. \tag{2.26}$$

Here dots over a quantity mean the partial derivative with respect to time, and the dyadic notation is used. (See Chapter 1 of Ref. 2.3.) Thus ∇ is the vector differential operator and $\underline{\underline{I}}$ is the unitary dyadic.

Over the surface, s, of a body (Fig. 2.1) with volume, v, the action of external sources is described in terms of forces or geometrical constraints, or more precisely, by means of prescribing tractions $\mathbf{T}^{(n)}$ or displacements $\mathbf{u}$. Thus the determination of the deformation of an elastic body can be formulated as a boundary value problem.

For a homogeneous, isotropic elastic body with volume v and surface $s = s_1 + s_2$, given body force $\rho\mathbf{f}$ in v, surface traction $\bar{\mathbf{T}}^{(n)}$ over s_1 with normal $\mathbf{n}$ and displacement $\bar{\mathbf{u}}$ over s_2, determine the displacement $\mathbf{u}(x_i, t)$ in v that satisfies the equations

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} + \rho\mathbf{f} = \rho\ddot{\mathbf{u}} \quad \text{in } v \tag{2.26}$$

with

$$\mathbf{T}^{(n)} = [\lambda(\nabla \cdot \mathbf{u})\underline{\underline{I}} + \mu(\nabla\mathbf{u} + \mathbf{u}\nabla)] \cdot \mathbf{n} = \bar{\mathbf{T}}^{(n)} \quad \text{on } s_1 \tag{2.27a}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } s_2. \tag{2.27b}$$

In addition, since the displacements are reckoned from some conveniently chosen time which is taken as $t = 0$, the displacement $\mathbf{u}_o$ and velocity $\dot{\mathbf{u}}_o$ at time $t = 0$, known as the *initial conditions*, must also be prescribed. Thus to the above equations, we add

$$\left.\begin{aligned}\mathbf{u}(x_i,0) &= \mathbf{u}_o\\ \dot{\mathbf{u}}(x_i,0) &= \dot{\mathbf{u}}_o\end{aligned}\right\} \quad \text{in } v. \tag{2.28}$$

This completes the mathematical formulation of the dynamic problems of elasticity.

As assured by the uniqueness theorem, if a solution were found which satisfied Eqs. (2.26), (2.27), and (2.28), it then would be the one and only solution. Dispensing the complete theorem and proof, we merely note the sufficient conditions for a unique solution (Chapter 7 of Ref. 2.4):

(a) Specification of the initial displacement and velocity throughout the body;

(b) Specification, at each and every point of the surface of any one of the eight combinations formed by choosing one member of each of the three products:

$$\sigma_{nn}u_n, \qquad \sigma_{ns}u_s, \qquad \sigma_{nt}u_t,$$

where n, s, and t indicate three mutually perpendicular directions and n is normal to the surface.

With the above conditions satisfied, a solution found for the boundary value problem which is formulated by equations (2.26) through (2.28) is ensured to be unique. The above theorem also suggests that boundary conditions other than (2.27) can be used. For instance, along a surface normal to the coordinate axis x_1, we can specify $u_1 = \sigma_{12} = \sigma_{13} = 0$, a condition sometimes called the rigid-smooth boundary. Such a boundary condition seems to be further removed from reality than the condition for a rigid surface (2.27b) or a traction-free (stress-free) surface (2.27a). However, they are encountered occasionally in practice when one is not sure of the real physical bounding conditions at the surface.

2.5. *Reduction to Wave Equations*

In the absence of body force, the displacement equation of motion is

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} = \rho\ddot{\mathbf{u}}. \tag{2.29}$$

According to the Helmholtz theorem, any vector field can be expressed as the sum of the gradient of a scalar field φ and the curl of a vector field $\mathbf{\Psi}$:

$$\mathbf{u} = \nabla\varphi + \nabla \times \mathbf{\Psi}, \qquad \nabla \cdot \mathbf{\Psi} = 0. \tag{2.30}$$

We call the φ and $\mathbf{\Psi}$ the scalar and vector *displacement potentials* respectively. Substitution of the above to (2.29) leads to

$$\nabla[(\lambda + 2\mu)\nabla^2\varphi - \rho\ddot{\varphi}] + \nabla \times [\mu\nabla^2\mathbf{\Psi} - \rho\ddot{\mathbf{\Psi}}] = 0,$$

which is satisfied if

$$\begin{aligned} c_p^2\nabla^2\varphi &= \ddot{\varphi}, \qquad & c_p^2 &= (\lambda + 2\mu)/\rho,\\ c_s^2\nabla^2\mathbf{\Psi} &= \ddot{\mathbf{\Psi}}, \qquad & c_s^2 &= \mu/\rho. \end{aligned} \tag{2.31}$$

It is seen that φ and $\mathbf{\Psi}$ satisfy a scalar and a vector wave equation respectively.

Since the wave equations are much simpler than the original equations of motion, solutions for $\mathbf{u}$ will be constructed from (2.30) in which the potentials satisfy the wave equations (2.31) and the boundary and initial conditions. A question arises as to whether every solution of (2.31) is included in the above procedure. This is answered by the completeness theorem (Ref. 2.5) stating that every solution of (2.29) admits a decomposition of (2.30) with φ and $\boldsymbol{\Psi}$ satisfying the equations (2.31).

The completeness theorem also assures that there are only two types of waves in an elastic solid, the one given by φ propagating with wave speed c_p, and the other given by $\boldsymbol{\Psi}$, which propagates with a speed c_s. Since $\lambda > 0$ and $\mu > 0$, c_p is always larger than c_s.

The ratio of these two wave speeds is a function of Poisson's ratio, ν, of the material only. Because of its frequent occurrence in elastodynamics, we denote it by $\varkappa$ with

$$\varkappa = \frac{c_p}{c_s} = \left(\frac{\lambda + 2\mu}{\mu}\right)^{\frac{1}{2}} = \left(\frac{2 - 2\nu}{1 - 2\nu}\right)^{\frac{1}{2}}. \tag{2.32}$$

The constant $\varkappa$ may be treated as another material constant in lieu of ν.

Because of its faster speed, the wave arising from $\nabla\varphi$ is called the *primary wave* (*P-wave*). The one from $\nabla \times \boldsymbol{\Psi}$ is called the *secondary wave* (*S-wave*). Since each type of wave can be identified by other physical characteristics, primary waves are known also as *dilatational*, *irrotational*, *compressional*, *longitudinal* waves, etc. The corresponding names for secondary waves are *rotational*, *equivoluminal*, *shear*, *transverse* waves, etc. In the ensuing discussion, the two types of wave will be identified simply as P-wave (Pressure or Primary wave) and S-wave (Shear or Secondary wave).

2.6. *Plane Waves*

In a solid, plane waves are represented by

$$u_i = A_i f(x_k \nu_k - ct),$$

or (2.33)

$$\mathbf{u} = \mathbf{A} f(\mathbf{r} \cdot \boldsymbol{\nu} - ct),$$

where $\mathbf{A}$ is the amplitude; f an arbitrary function; $\boldsymbol{\nu}$, the wave normal, which indicates the direction of wave propagation; and c is the wave speed. Not every c and every amplitude vector $\mathbf{A}$ are feasible solutions of (2.29). Substitution shows that only when

$$(\lambda + \mu)(\mathbf{A} \cdot \boldsymbol{\nu})\boldsymbol{\nu} + (\mu - \rho c^2)\mathbf{A} = 0$$

does the form given by (2.33) represent plane waves in elastic solids. The equation directly above is satisfied if

$$\begin{aligned} &(1) \quad \mathbf{A} = |\mathbf{A}|\boldsymbol{\nu} \quad \text{and} \quad c^2 = (\lambda + 2\mu)/\rho = c_p^2, \\ &(2) \quad \mathbf{A} \cdot \boldsymbol{\nu} = 0 \quad \text{and} \quad c^2 = \mu/\rho = c_s^2. \end{aligned} \tag{2.34}$$

The first is a plane P-wave with the displacement vector parallel to the wave normal; the second is a plane S-wave where the displacement vector is always at right angles to the wave normal.

Instead of (2.33) and the condition (2.34), we can also represent plane elastic waves in terms of displacement potentials. A P-wave is given by

$$\begin{aligned} \varphi &= \varphi_o f(\mathbf{r} \cdot \boldsymbol{\nu} - c_p t), \\ \boldsymbol{\Psi} &= 0, \end{aligned} \tag{2.35}$$

with corresponding displacements

$$\mathbf{u}_p = \nabla\varphi = \varphi_o \mathbf{v} f'(\mathbf{r}\cdot\mathbf{v} - c_p t), \qquad (2.36)$$

where prime indicates differentiation with respect to the argument.

Shear waves can be represented by

$$\varphi = 0,$$

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}_o g(\mathbf{r}\cdot\mathbf{v} - c_s t),$$

where $\boldsymbol{\Psi}_o$ is any constant vector perpendicular to $\mathbf{v}$. Let $\mathbf{p}$, a unit vector, indicate the polarization of a shear wave. We can write $\boldsymbol{\Psi}_o = \psi_o \mathbf{p} \times \mathbf{v}$ and represent the shear wave by

$$\varphi = 0, \qquad (2.37)$$
$$\boldsymbol{\Psi} = \psi_o(\mathbf{p}\times\mathbf{v}) g(\mathbf{r}\cdot\mathbf{v} - c_s t),$$

where ψ_o is a scalar coefficient. The potentials in (2.37) yield the displacement vector

$$\mathbf{u}_s = \nabla\times\boldsymbol{\Psi} = \psi_o \mathbf{p} g'(\mathbf{r}\cdot\mathbf{v} - c_s t), \qquad \mathbf{p}\cdot\mathbf{v} = 0. \qquad (2.38)$$

There is no ambiguity about the polarization of a plane P-wave, as the displacement vector $\mathbf{u}_p$ is always parallel to the wave normal $\mathbf{v}$. However, unless $\mathbf{p}$ in (2.38) is specified, the polarization of a shear wave is uncertain because $\mathbf{u}_s$ may be along any one of the infinite unit vectors normal to $\mathbf{v}$ (Fig. 2.2). As a matter of convenience, we choose, in a medium, a straight line as the *vertical* axis (x_2-axis in Fig. 2.2) and refer to any plane perpendicular to the straight line as the horizontal plane. Shear waves with polarization parallel to

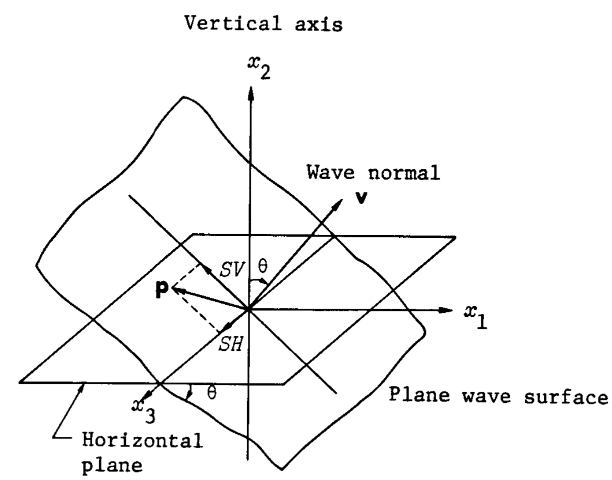

Fig. 2.2. Polarization of S-Wave

the horizontal x_3-axis are called SH waves; the ones with polarization parallel to a vertical plane (x_1-x_2 plane) are SV waves. This reference system is very convenient in dealing with seismic waves as the ground surface provides a natural horizontal base. For other problems, we can judiciously choose a particular direction as the vertical axis, and the rest of the axes follow.

The functions f and g in the above are arbitrary functions, including the generalized functions. Familiar examples are the unit step function

$$h(\xi) = \begin{cases} 0, & \xi < 0, \\ 1, & \xi > 0, \end{cases}$$

and the delta function

$$\delta(\xi) = h'(\xi).$$

If in (2.36), $f'(\xi) = h(\xi)$ and

$$\mathbf{u}_p = \varphi_o \mathbf{v} h(c_p t - \mathbf{r} \cdot \mathbf{v}), \tag{2.39}$$

(2.39) represents a step plane P-wave. At a given position $\mathbf{r} = \mathbf{r}_o$, there is no disturbance for $t < (\mathbf{r}_o \cdot \mathbf{v}/c_p)$. At time $t = \mathbf{r}_o \cdot \mathbf{v}/c_p$, the displacement suddenly jumps to the magnitude φ_o and remains constant thereafter. If $f'(\xi)$ in (2.36) is a delta function, the displacement at the station $\mathbf{r}_o$ increases suddenly to a large value at $t = \mathbf{r}_o \cdot \mathbf{v}/c_p$ and dies out as the pulse passes.

Another function of special interest is the harmonic function $\exp(\pm i\omega t)$ where ω is the *circular frequency* (radians per unit time). It is understood that whenever such a function is used, only the real or the imaginary part represents the motion. Thus, if in (2.35) we let

$$\varphi = \varphi_o e^{ik(\mathbf{r}\cdot\mathbf{v}-c_p t)} = \varphi_o e^{ik\Theta}, \qquad \boldsymbol{\Psi} = 0, \tag{2.40}$$

$$kc_p = \omega, \qquad k\mathbf{v} = \mathbf{k}, \qquad \Theta = \mathbf{r}\cdot\mathbf{v} - c_p t,$$

the displacement has the form

$$\mathbf{u}_p = ik\varphi_o \mathbf{v} e^{ik\Theta}, \tag{2.41}$$

where the coefficient $\varphi_o = a + ib$ is taken as a complex number. The actual motion is represented by either the real part

$$\mathbf{u}_p = -\mathbf{v}k(a \sin k\Theta + b \cos k\Theta) = -\mathbf{v}k\sqrt{a^2 + b^2} \cos(k\Theta - \boldsymbol{\epsilon}_1) \tag{2.42a}$$

or the imaginary part

$$\mathbf{u}_p = \mathbf{v}k(a \cos k\Theta - b \sin k\Theta) = \mathbf{v}k\sqrt{a^2 + b^2} \cos(k\Theta + \boldsymbol{\epsilon}_2) \tag{2.42b}$$

with $\boldsymbol{\epsilon}_1 = \tan^{-1}(a/b)$ and $\boldsymbol{\epsilon}_2 = \tan^{-1}(b/a)$. Since

$$k\Theta = k\mathbf{r}\cdot\mathbf{v} - \omega t = \mathbf{k}\cdot\mathbf{r} - \omega t,$$

the motion is simple harmonic in time and sinusoidal in space. We call Θ the *phase* of the wave, $\mathbf{k}$ the *vector wave number*, and $\boldsymbol{\epsilon}$ the *phase constant*. The wave number is related to the *wave length* λ by $k = 2\pi/\lambda$ and to the circular frequency ω by $k = \omega/c_p$ for the P-wave. As can be seen from (2.41) and (2.42) we could use the sine and cosine functions directly, in lieu of the exponential form, to represent the waves.

Wave propagations that are represented by harmonic function $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)](-\infty < t < \infty)$ as in (2.41) are designated as the *steady state*. The motion continues, and no time can be reckoned for when the motion started, nor for when it is to end. If the body is at rest before a certain time $t = t_o$ and the motion is initiated at $t = t_o$, the waves are in the *transient state*. Wave motion represented by (2.39) is clearly a transient one; so is the motion

$$\mathbf{u}_p = \begin{cases} \varphi_o \mathbf{v} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, & t > |\mathbf{r}|/c_p, \\ 0, & t < |\mathbf{r}|/c_p. \end{cases}$$

The latter, although a sinusoidal motion, does not start at a given position until time $t = t_o = |\mathbf{r}|/c_p$.

In Section 4, we shall discuss these two types of motion in detail.

It is sufficient to note here that they are related through the principle of superposition. Knowing the steady state wave propagation, the transient phenomena can be determined by superposing steady waves of all frequencies.

2.7. *Equations in Orthogonal Curvilinear Coordinates*

From Eqs. (2.30) and (2.31) it is seen that the displacement field of an elastic solid in motion can be represented by a scalar potential φ which satisfies a scalar wave equation, and by a vector potential $\boldsymbol{\Psi}$ satisfying a vector wave equation. The vector wave equation is actually a composition of three equations for the three components of $\boldsymbol{\Psi}$. Except in Cartesian coordinates, all three components may occur in all three equations when they are expressed in terms of orthogonal curvilinear coordinates. The coupling of all three unknown components of $\boldsymbol{\Psi}$ in three equations causes immense difficulties in tracking the solution of the vector wave equation.

Even if the solutions of the simultaneous equations are obtainable, there is still the difficulty of meeting the boundary conditions as the components of the potential $\boldsymbol{\Psi}$ must be combined with φ to yield displacements and stresses which are the usual quantities specified on the boundary of an elastic solid. If, however, the boundary is one of the orthogonal curvilinear coordinate surfaces, some simplification is possible. A general approach is to decompose a vector wave field into three parts, with each part determinable by a scalar wave function. (See Chapter 13 of Ref. 2.3.)

Let ξ_1, ξ_2, and ξ_3 be the three orthogonal curvilinear coordinates,

with scale factors h_1, h_2, and h_3 defined by

$$ds^2 = \sum_{i=1}^{3} (dx_i)^2 = \sum_{i=1}^{3} (h_i d\xi_i)^2,$$

where x_i are the Cartesian coordinates and ds is the length of a line element. When the transformation of the coordinates is specified by

$$x_i = x_i(\xi_1, \xi_2, \xi_3), \qquad i = 1,2,3,$$

or its inverse

$$\xi_i = \xi_i(x_1, x_2, x_3),$$

the scale factors can be computed by

$$h_j^2 = \left(\frac{\partial x_1}{\partial \xi_j}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_j}\right)^2 + \left(\frac{\partial x_3}{\partial \xi_j}\right)^2 = \left[\left(\frac{\partial \xi_j}{\partial x_1}\right)^2 + \left(\frac{\partial \xi_j}{\partial x_2}\right)^2 + \left(\frac{\partial \xi_j}{\partial x_3}\right)^2\right]^{-1}.$$

The direction of each coordinate curve is indicated by a unit vector $\mathbf{e}_i (i = 1,2,3)$. For scattering problems, one or several of the coordinate surfaces may form the boundary of a scatterer (Fig. 2.3).

The dilatational part of the displacement $\mathbf{u}$ is given by the gradient of a scalar potential φ with

$$\mathbf{u} \text{ (dilatational)} = \boldsymbol{\iota} = \nabla\varphi \tag{2.43}$$

and the rotational part by $\nabla \times \boldsymbol{\Psi}$ with $\nabla \cdot \boldsymbol{\Psi} = 0$. On account of the condition of zero divergence, only two of the three components of $\boldsymbol{\Psi}$ are independent. Since $\boldsymbol{\Psi}$ gives rise to a shear wave, the resolution of a plane shear wave into SV-wave and SH-wave suggests that $\boldsymbol{\Psi}$ may be decomposed into two parts: One is a vector along a preferred direction,

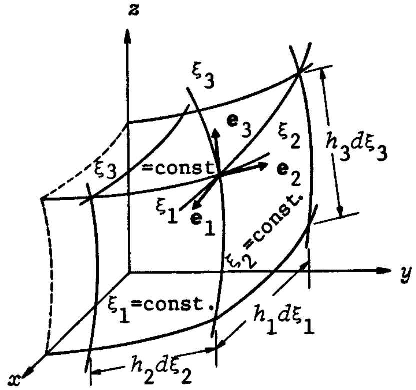

Fig. 2.3. Curvilinear Coordinate System (ξ_1, ξ_2, ξ_3)

say $\mathbf{e}_3$, and the other is at right angles to the first vector. Thus we set

$$\boldsymbol{\Psi} = \mathbf{e}_3(w\psi) + \ell\nabla \times (\mathbf{e}_3 w\chi),$$

where $\psi(\xi_1,\xi_2,\xi_3, t)$ and $\chi(\xi_1,\xi_2,\xi_3, t)$ are two unspecified scalar functions, $w(\xi_3)$ is a function of coordinate ξ_3, and ℓ is a scalar factor having the dimension length. The factor ℓ is introduced to give χ the same dimension as ψ.

The two displacements corresponding to ψ and χ are represented by $\mathbf{M}$ and $\mathbf{N}$ respectively, with

$$\begin{aligned} \mathbf{u}\ (\text{rotational}) &= \mathbf{M} + \mathbf{N} \\ &= \nabla \times \boldsymbol{\Psi} \\ &= \nabla \times (\mathbf{e}_3 w\psi) + \ell\nabla \times \nabla \times (\mathbf{e}_3 w\chi). \end{aligned} \tag{2.44}$$

It is clear that the first component $\mathbf{M} = \nabla \times (\mathbf{e}_3 w\psi) = \nabla(w\psi) \times \mathbf{e}_3$ is

perpendicular to $\mathbf{e}_3$ and is tangent to the surface ξ_3 = constant. For the case of plane wave propagation Cartesian coordinates are used. If $\mathbf{e}_3$ is taken in the direction of wave normal, the coordinate surfaces ξ_3 = constant are then the wave fronts, and ψ and χ give rise to the plane SV- and SH-wave respectively. (See Chapter II, Section 2.) However, when the curvilinear coordinates are used for the general wave propagation, the coordinate surfaces are not necessarily the wave fronts. Thus the shear wave displacement may or may not be tangent to the coordinate surfaces.

As a displacement potential each part of $\boldsymbol{\Psi}$ must be a solution of the vector wave equation (2.31). In other words, ψ, χ, and w should be chosen so as to satisfy the following equations:

$$\begin{aligned} c_s^{\,2}\nabla^2(\mathbf{e}_3 w\psi) &= \mathbf{e}_3 w\ddot{\psi}, \\ c_s^{\,2}\nabla^2[\nabla \times (\mathbf{e}_3 w\chi)] &= \nabla \times (\mathbf{e}_3 w\ddot{\chi}). \end{aligned} \tag{2.45}$$

The restriction on the choice of ψ could be somewhat relaxed if we recall that in the derivation of (2.31), the condition that $\boldsymbol{\Psi}$ must meet is actually

$$\nabla \times [\mu\nabla^2\boldsymbol{\Psi} - \rho\ddot{\boldsymbol{\Psi}}] = 0.$$

Hence, instead of (2.45), the ψ could be chosen to satisfy

$$\nabla \times [c_s^{\,2}\nabla^2(\mathbf{e}_3 w\psi) - \mathbf{e}_3 w\ddot{\psi}] = 0. \tag{2.46}$$

We proceed first with the determination of ψ and note that for any vector field $\mathbf{F}$,

$$\nabla^2\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) - \nabla \times \nabla \times \mathbf{F}.$$

Applying the well-known formulas of vector calculus in curvilinear coordinates, we find

$$\nabla\nabla\cdot(\mathbf{e}_3 w\psi) = \nabla\left[\frac{1}{h_1h_2h_3}\frac{\partial}{\partial\xi_3}(h_1h_2w\psi)\right],$$

$$\nabla\times\nabla\times(\mathbf{e}_3w\psi) = \frac{\mathbf{e}_1}{h_2h_3}\frac{\partial}{\partial\xi_3}\left[\frac{h_2}{h_1h_3}\frac{\partial(h_3w\psi)}{\partial\xi_1}\right] + \frac{\mathbf{e}_2}{h_3h_1}\frac{\partial}{\partial\xi_3}\left[\frac{h_1}{h_2h_3}\frac{\partial(h_3w\psi)}{\partial\xi_2}\right]$$
$$- \frac{\mathbf{e}_3}{h_1h_2}\left\{\frac{\partial}{\partial\xi_1}\left[\frac{h_2}{h_1h_3}\frac{\partial(h_3w\psi)}{\partial\xi_1}\right] + \frac{\partial}{\partial\xi_2}\left[\frac{h_1}{h_2h_3}\frac{\partial(h_3w\psi)}{\partial\xi_2}\right]\right\}.$$

If h_3 is constant and h_2/h_1 is independent of the coordinate ξ_3, the above expressions can be simplified considerably with the result

$$\nabla^2(\mathbf{e}_3w\psi) = \frac{1}{h_3}\nabla\left[\frac{w\psi}{h_1h_2}\frac{\partial(h_1h_2)}{\partial\xi_3}\right]$$
$$+ \mathbf{e}_3\left[\nabla^2(w\psi) - \frac{1}{h_1h_2h_3^2}\frac{\partial(w\psi)}{\partial\xi_3}\frac{\partial(h_1h_2)}{\partial\xi_3}\right]. \qquad (2.47)$$

It is comfortable to note that in addition to the Cartesian coordinate system, all the cylindrical coordinate systems (circular, elliptical, and parabolic), the spherical coordinate and the conical coordinate systems also belong to this category.

For the three cylindrical coordinates, h_1h_2 is also independent of the ξ_3 coordinate. We thus set w = constant and reduce Eq. (2.47) to

$$\nabla^2(\mathbf{e}_3w\psi) = w\mathbf{e}_3\nabla^2\psi.$$

It is seen that the vector wave equation (2.45) is satisfied if

$$c_s^2\nabla^2\psi = \ddot{\psi}, \qquad (2.48)$$

which is a scalar wave equation for the wave potential function $\psi(\xi_1, \xi_2, \xi_3, t)$.

For spherical and conical coordinates, $h_1h_2 = \xi_3^2 f(\xi_1,\xi_2)$, the choice of w = constant is to no advantage. Substituting the following result

$$\nabla^2(w\psi) = w\nabla^2\psi + 2(\nabla w)\cdot(\nabla\psi) + \psi\nabla^2 w$$

into (2.47), we obtain

$$\nabla^2(\mathbf{e}_3w\psi) = \frac{1}{h_3}\nabla\left[\frac{w\psi}{h_1h_2}\frac{\partial(h_1h_2)}{\partial\xi_3}\right]$$
$$+ \mathbf{e}_3\left\{w\nabla^2\psi + \frac{1}{h_3^2}\left[2\frac{\partial w}{\partial\xi_3}\frac{\partial\psi}{\partial\xi_3} + \psi\frac{\partial^2 w}{\partial\xi_3^2} - \frac{w}{h_1h_2}\frac{\partial\psi}{\partial\xi_3}\frac{\partial(h_1h_2)}{\partial\xi_3}\right]\right\}$$
$$(2.49)$$

Of the coefficients of $\mathbf{e}_3$ the terms inside the brackets will cancel each other if $w(\xi_3)$ is linear in ξ_3. Setting $w_3 = \xi_3$, we obtain

$$\nabla^2(\mathbf{e}_3w\psi) = (2/h_3)\nabla\psi + \mathbf{e}_3(w\nabla^2\psi).$$

The first term with grad ψ need not concern us here because curl(grad ψ) = 0. Hence if ψ is a solution of the scalar wave equation as in (2.48), the condition (2.46) is satisfied.

In either of the above two cases, the task of finding the first part of the potential $\mathbf{\Psi}$ is reduced to solving the scalar wave equation (2.48). Admittedly, this procedure works for only six systems of curvilinear coordinates.

The second scalar potential χ can be determined in exactly the same manner. It is to be a solution of the scalar wave equation

$$c_s^2\nabla^2\chi = \ddot{\chi}. \tag{2.50}$$

In summary, the displacement vector $\mathbf{u}$ may be resolved into a dilatation part (P-wave) $\mathbf{L} = \nabla\varphi$ and a rotational part $\nabla \times \boldsymbol{\Psi}$ (S-wave) where φ satisfies a scalar wave equation and $\boldsymbol{\Psi}$ a vector wave equation. Out of the eleven coordinate systems (confocal quadric surfaces) for which the solution of a scalar wave equation is separable into product of three factors, each dependent on only one coordinate, there are only six coordinates for which the vector wave equations are separable. They are: Cartesian, circular cylindrical, elliptic cylindrical, parabolic cylindrical, spherical, and conical coordinates. For those systems with coordinates $\xi_i (i = 1, 2, 3)$ and unit vectors $\mathbf{e}_i$, the rotational part may be decomposed further into two components $\mathbf{M}$ and $\mathbf{N}$ with

$$\mathbf{u} = \mathbf{L} + \mathbf{M} + \mathbf{N} \tag{2.51}$$

and

$$\mathbf{L} = \nabla\varphi,$$

$$\mathbf{M} = \nabla \times (\mathbf{e}_3\omega\psi) = \nabla(\omega\psi) \times \mathbf{e}_3, \tag{2.52}$$

$$\mathbf{N} = \ell\nabla \times \nabla \times (\mathbf{e}_3\omega\chi) = \ell\nabla\left[\frac{\partial(\omega\chi)}{\partial\xi_3}\right] - \ell\mathbf{e}_3\omega\nabla^2\chi$$

where ℓ is a scalar factor with length dimension and φ, ψ, χ satisfy the following scalar wave equations [Eqs. (2.31), (2.48), and (2.50)]:

$$\left.\begin{aligned} c_p^2\nabla^2\varphi &= \ddot{\varphi}, \\ c_s^2\nabla^2\psi &= \ddot{\psi}, \\ c_s^2\nabla^2\chi &= \ddot{\chi}. \end{aligned}\right\} \tag{2.53}$$

The vector $\mathbf{M}$ is always perpendicular to the unit vector $\mathbf{e}_3$, and $\mathbf{N}$ is at right angles to $\mathbf{M}$ when $\partial(\omega\chi)/\partial\xi_3$ is proportional to $\omega\psi$.

For the Cartesian system, $\mathbf{e}_3$ may be directed along any one of the coordinates and $\omega(\xi_3)$ is taken as 1.

For the three cylindrical coordinate systems, $\mathbf{e}_3$ is taken along the axis of the cylinder and $\omega = 1$.

For the spherical or conical coordinate systems $\mathbf{e}_3$ is directed toward the radial direction and $\omega = \xi_3$.

For the other five coordinate systems, this resolution fails to apply. However, for problems with axial symmetry, one of the displacement components vanishes and the rotational part of a displacement vector is determined by a single wave potential. Hence the number of separable coordinate systems for a vector wave field is increased to nine with the addition of parabolic, prolate spheroidal, and oblate spheroidal coordinates. Solutions for vector wave equations in prolate and oblate spheroidal coordinates and with axial symmetry are discussed in Refs. (2.6) and (2.7).

3. TWO-DIMENSIONAL APPROXIMATIONS OF ELASTICITY

IN THE MOST GENERAL CASE, the displacement components in Eq. (2.26) are functions of time t and three spatial coordinates x_1, x_2, and x_3.

The equations, even when they can be reduced to wave equations, have four independent variables. Their solutions are indeed very difficult.

In many applications, the elastic body of concern has a characteristic length in one direction either much longer or much shorter than the others. Furthermore the cross sections along that direction may be uniform. Typical examples are the footing of a wall resting on a large foundation, a long tunnel under a flat surface, and a thin plate with constant thickness. When bodies with such geometry are subjected to a special distribution of external forces, various approximations can be made of the displacement components to simplify the equations of motion.

Two types of geometry are of special interest. One is that of a bulky solid with uniform cross section along one direction. A long cylindrical underground tunnel is a familiar example (Fig. 3.1). Problems of this type can be classified as *anti-plane strain* or *plane strain*, according to how the external forces are applied. When a distributed force is applied parallel to the lengthwise direction (x_3 in Fig. 3.1) such that the dominant displacement is in that direction and has constant magnitude along the lengthwise axis, the problem falls into the category of anti-plane strain. If the force is applied perpendicularly to the characteristic length, and distributed uniformly in the lengthwise direction, there is then functionally no displacement along that direction. It is a problem of plane-strain.

The second type of geometry is that of a thin plate of uniform thickness. The plate could be perforated, or it may have cracks. Plates with external load applied parallel to the mid-surface are

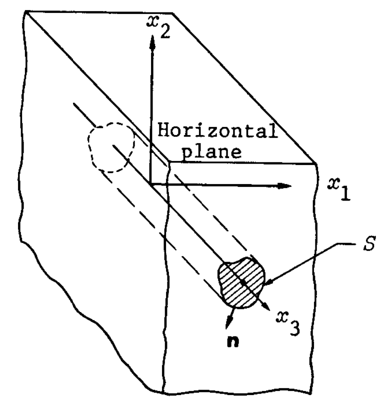

Fig. 3.1. Geometry of Plane Strain and Anti-Plane Strain

grouped as presenting problems of *generalized plane stress*. When the forces are applied transversely to the plate, the problem is one of *bending*.

These four types of problems will be defined more precisely in the following subsections. A two-dimensional approximation of each problem is given therein. Whether an actual problem can be treated by one of these approximations will become clear after the assumptions are made evident in each formulation.

3.1. *Anti-Plane Strain*

Refer a bulky elastic body to a Cartesian coordinate system x_1, x_2, x_3, and let x_3 be the special direction along which the cross-sectional areas of the body are constant (Fig. 3.1). Anti-plane strain is defined as

$$u_1 = 0,$$

$$u_2 = 0, \tag{3.1}$$

$$u_3 = w(x_1, x_2, t).$$

From (2.23), the stresses for an isotropic elastic solid are

$$\sigma_{13} = \mu \frac{\partial w}{\partial x_1},$$

$$\sigma_{23} = \mu \frac{\partial w}{\partial x_2}, \tag{3.2}$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0.$$

The assumption (3.1) reduces the third of the equation of motion (2.26) to a single scalar wave equation

$$\mu \nabla^2 w(x_1, x_2, t) = \rho \frac{\partial^2 w}{\partial t^2}, \tag{3.3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

The other two equations are satisfied identically. If the elastic body is bounded by the surface S defined by

$$f(x_1, x_2) = 0 \tag{3.4}$$

the boundary condition for a rigid surface will be

$$w = 0 \quad \text{on} \quad S, \tag{3.5a}$$

whereas for a traction-free surface, with outernormal $\mathbf{n}$, the boundary condition is

$$n_1\sigma_{13} + n_2\sigma_{23} = \mu\left[\cos\,(x_1,n)\,\frac{\partial w}{\partial x_1} + \cos\,(x_2,n)\,\frac{\partial w}{\partial x_2}\right] = 0 \quad \text{on } S. \tag{3.5b}$$

Although the equations above are formulated in terms of Cartesian coordinates, their generalization to cylindrical coordinates is immediate. For this generalization, x_1 and x_2 are considered as plane curvilinear coordinates, perpendicular to the x_3-axis, and equation (3.1) still holds. There are still only two nonvanishing stress components, σ_{13} and σ_{23}. Their relation to displacement w should then be calculated from (2.25) where ∇ is a two-dimensional gradient operator for the particular plane curvilinear coordinates. The Laplacian operator in the scalar wave equation should also be converted accordingly.

Under the assumption of anti-plane strain, the dilatation $\nabla \cdot \mathbf{u}$ is zero, and the waves are rotational (S-waves). Because the displacement vector of the wave is always parallel to the x_3-axis, which for convenience can be taken as lying on a *horizontal* plane, we shall call the waves of anti-plane strain *SH waves*. Strictly speaking, the name manifests itself only when there is a direction which can be clearly labelled as horizontal.

The SH wave so defined is mathematically analogous to sound waves in air. If φ denotes a velocity potential such that the velocity vector of sound wave $\mathbf{v} = \nabla\varphi$, then φ satisfies also a scalar wave equation

$$c^2\nabla^2\varphi = \ddot{\varphi}, \qquad c^2 = (dp/d\rho), \tag{3.6}$$

where the pressure p and condensation s (the relative change of density) are related to the potential by

$$p = -\rho \frac{\partial \varphi}{\partial t}, \qquad s = \frac{p}{\rho c^2}. \tag{3.7}$$

Thus w in (3.3) corresponds to the velocity potential φ; a rigid surface in anti-plane strain ($w = 0$) corresponds to an opening of an acoustic conduit where the pressure is taken to be zero. A stress-free surface would be an unmovable wall to a sound wave, at which the velocity would be zero.

It is to be emphasized that the analogy of sound waves to SH waves is purely a mathematical one. Physically, the sound wave is of the dilatation type, and for plane waves, the motion of air particles is in the direction of propagation. It is a longitudinal wave. The plane SH wave is a transverse wave.

3.2. *Plane Strain*

Referring to the same geometry as illustrated in Fig. 3.1, plane strain is defined as

$$\begin{aligned} u_1 &= u(x_1,x_2,t), \\ u_2 &= v(x_1,x_2,t), \\ u_3 &= 0. \end{aligned} \tag{3.8}$$

Because of this assumption, the shearing stresses along the x_3-axis σ_{13}, σ_{23} are always zero, and the normal stress σ_{33} is related to the others by

$$\sigma_{33} = -\nu(\sigma_{11} + \sigma_{22}).$$

In Cartesian coordinates, the other nonvanishing stresses are related to displacements by

$$\sigma_{\alpha\beta} = \lambda \frac{\partial u_\gamma}{\partial x_\gamma} \delta_{\alpha\beta} + \mu\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right), \qquad \alpha,\beta,\gamma = 1,2. \tag{3.9}$$

They resemble (2.23) in general form except that the indices only take the value of 1 and 2. Similarly, the equations of motion reduce to two scalar equations:

$$(\lambda + \mu) \frac{\partial^2 u_\alpha}{\partial x_\alpha \partial x_\beta} + \mu \frac{\partial^2 u_\beta}{\partial x_\alpha \partial x_\alpha} + \rho f_\beta = \rho \frac{\partial^2 u_\beta}{\partial t^2}, \qquad \alpha,\beta = 1,2; \tag{3.10}$$

the third one vanishes.

The decomposition of the displacement vector into two potentials still applies. The condition (3.8) is satisfied if in (2.30)

$$\boldsymbol{\Psi} = \psi \mathbf{e}_3.$$

Thus, instead of (2.30) and (2.31), we have for a plane strain:

$$\mathbf{u} = \nabla\varphi(x_1,x_2,t) + \nabla \times [\mathbf{e}_3\psi(x_1,x_2,t)] \tag{3.11}$$

and

$$\begin{aligned} c_p^2\nabla^2\varphi &= \ddot{\varphi}, \qquad c_p^2 = (\lambda + 2\mu)/\rho, \\ c_s^2\nabla^2\psi &= \ddot{\psi}, \qquad c_s^2 = \mu/\rho. \end{aligned} \tag{3.12}$$

The above equations, together with the stress-displacement relation

$$\underline{\underline{\sigma}} = \lambda(\nabla \cdot \mathbf{u})\underline{\underline{I}} + \mu(\nabla\mathbf{u} + \mathbf{u}\nabla), \tag{3.13}$$

complete the field equations for plane strain. All quantities in (3.11) to (3.13) are independent of x_3 coordinates and $\mathbf{u}$ and $\underline{\underline{\sigma}}$ are two-dimensional vector and dyadics, respectively.

From (3.11), we find that the shear wave derived from the potential ψ in plane strain has displacements

$$u_1 = \partial\psi/\partial x_2, \qquad u_2 = -\partial\psi/\partial x_1.$$

The displacement vector is always transverse to the x_3-axis. We shall call the shear wave defined by (3.11) and (3.12) an *SV-wave* in the following studies.

3.3. *Generalized Plane Stress*

For an elastic plate under the action of external forces applied parallel to the mid-plane (Fig. 3.2), calculations of the displacement

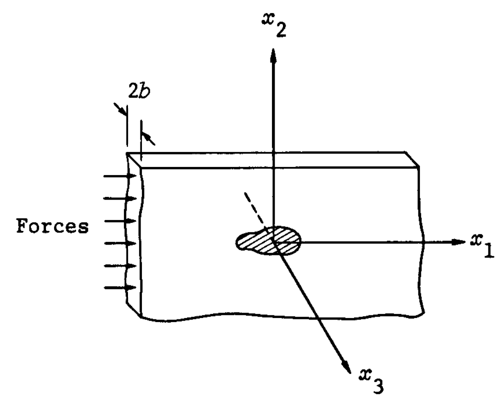

Fig. 3.2. Geometry of Generalized Plane Stress

or stress variation across the plate thickness are very difficult. In a case where the plate thickness is small in comparison with the wavelength and other dimensions, a set of two-dimensional equations can be derived for the average values of displacement and stresses across the thickness. The appropriate equations were obtained by Poisson (1829)(1.12) and Cauchy (1828)(3.1) by expanding the displacement components into power series of the thickness coordinates and truncating the series at various orders of approximation. Given here is a brief account of the equations with a derivation similar to that by Filon (1903)(3.2) for *generalized plane stress* in elastostatics.

Let the x_1-x_2 coordinate plane be the mid-surface of the plate, and denote the average stress and displacements by $\bar{\sigma}_{ij}$, $\bar{u}_j$

$$\bar{u}_j(x_1,x_2,t) = \frac{1}{2b}\int_{-b}^{b} u_j(x_1,x_2,x_3,t)dx_3, \tag{3.14}$$

$$\bar{\sigma}_{ij}(x_1,x_2,t) = \frac{1}{2b}\int_{-b}^{b} \sigma_{ij}(x_1,x_2,x_3,t)dx_3, \tag{3.15}$$

where $2b$ is the thickness of the plate. Since the plate is thin and free from stresses at surface $x_3 = \pm b$, we assume that

$$\bar{\sigma}_{31} = \bar{\sigma}_{32} = \bar{\sigma}_{33} = 0. \tag{3.16}$$

In the stress equation of motion (2.12), if the same average procedure is performed, we have, with $\bar{f}_3 = 0$

$$\frac{\partial\bar{\sigma}_{\alpha\beta}}{\partial x_\alpha} + \rho\bar{f}_\beta = \rho\,\frac{\partial^2\bar{u}_\beta}{\partial t^2}, \qquad \alpha,\beta = 1,2. \tag{3.17}$$

Similar average processes are performed for the strain displacement relation, (2.5), and the stress-strain relation (2.20)

$$\bar{\sigma}_{ij} = \lambda\bar{\varepsilon}_{kk}\delta_{ij} + 2\mu\bar{\varepsilon}_{ij},$$
$$\bar{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right), \qquad i,j = 1,2,3. \tag{3.18}$$

Since $\bar{\sigma}_{33} = 0$ by assumption, the average normal strain $\bar{\varepsilon}_{33}$ is related to the other two by

$$\bar{\varepsilon}_{33} = -\frac{\lambda}{\lambda + 2\mu}(\bar{\varepsilon}_{11} + \bar{\varepsilon}_{22}). \tag{3.19}$$

Substitution of the above to (3.18) results in

$$\bar{\sigma}_{\alpha\beta} = \lambda'\bar{\varepsilon}_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu\bar{\varepsilon}_{\alpha t}$$

or

$$\bar{\sigma}_{\alpha\beta} = \lambda'\left(\frac{\partial\bar{u}_\gamma}{\partial x_\gamma}\right)\delta_{\alpha\beta} + \mu\left(\frac{\partial\bar{u}_\alpha}{\partial x_\beta} + \frac{\partial\bar{u}_\beta}{\partial x_\alpha}\right), \qquad \alpha,\beta,\gamma = 1,2, \tag{3.20}$$

where

$$\lambda' = \frac{2\lambda\mu}{\lambda + 2\mu} = \frac{2\mu\nu}{1 - \nu}. \tag{3.21}$$

Combining (3.20) with (3.17) gives

$$(\lambda' + \mu)\frac{\partial^2\bar{u}_\alpha}{\partial x_\alpha \partial x_\beta} + \mu\frac{\partial^2\bar{u}_\beta}{\partial x_\alpha \partial x_\alpha} + \rho\bar{f}_\beta = \rho\frac{\partial^2\bar{u}_\beta}{\partial t^2}, \qquad \alpha,\beta = 1,2. \tag{3.22}$$

With $\bar{u}_\alpha$, $\bar{\sigma}_{\alpha\beta}$ defined by Eqs. (3.14) tnrough (3.16), Eqs. (3.20) and (3.22) complete the generalized plane stress equation. All quantities in the above equations are independent of the x_3-coordinate.

Comparison of Eqs. (3.20) and (3.22) with Eqs. (3.9) and (3.10) for plane strain shows that these two formulations are mathematically identical, the only difference being that the material constant λ in plane strain is replaced by $\lambda' = 2\lambda\mu/(\lambda + 2\mu)$ in plane stress. Thus the solution for a long bulk solid can be converted to the one for a thin disk and vice versa, providing all other conditions are the same. From plane strain solution to plane stress solution, we simply replace λ by λ' and leave the constant μ unchanged. If a solution for a plane stress is known, changing λ to $2\lambda'\mu/(2\mu - \lambda')$ will convert it to the one for plane strain. Conversion of other elastic constants are summarized in Table 3.1.

Table 3.1

CONVERSION OF ELASTIC CONSTANTS IN PLANE ELASTICITY

Constant	Plane Strain to Plane Stress	Plane Stress to Plane Strain
λ	$\frac{2\lambda\mu}{\lambda + 2\mu}$	$\frac{2\lambda\mu}{2\mu - \lambda}$
μ	μ	μ
ν	$\frac{\nu}{1 + \nu}$	$\frac{\nu}{1 - \nu}$
E	$E\,\frac{1 + 2\nu}{(1 + \nu)^2}$	$\frac{E}{1 - \nu^2}$

Unlike the plane strain solution, the plane stress solution is formulated in terms of the average stresses and average displacements across the thickness of the plate, not in terms of the pertinent values at each field point. It may thus lead to erroneous results when the displacements vary sharply across the thickness. Such is the case when the plate vibrates longitudinally at high frequencies. A comparison with the results based on three-dimensional elasticity theory shows that the generalized plane stress approximation for a plate is valid when the frequency is less than $\pi c_s/2b$ and when the wavelength is longer than $2\pi b$.(3.3)

Under the plane stress assumption, the displacement, a two-dimensional vector, can be determined from two potentials φ and ψ as in (3.11). The field associated with ψ is still the SV wave, but that part from φ is no longer the P-wave. The waves of φ, still dilatational, travel with the speed

$$c_p' = [(\lambda' + 2\mu)/\rho]^{\frac{1}{2}} = [2\mu/\rho(1 - \nu)]^{\frac{1}{2}}. \qquad (3.23)$$

They are known as the *extensional waves* in plates.

3.4. *Bending of a Plate*

If the plate discussed in the previous subsection is subjected to external forces applied perpendicular to the plate, waves of an entirely different nature are generated. To analyze the dominant feature of the plate motion under such forces, we assume that (Fig. 3.3)

$$\begin{aligned} u_1(x_1,x_2,x_3,t) &= x_3\psi_1(x_1,x_2,t), \\ u_2(x_1,x_2,x_3,t) &= x_3\psi_2(x_1,x_2,t), \qquad (3.24) \\ u_3(x_1,x_2,x_3,t) &= w(x_1,x_2,t) \end{aligned}$$

and define

$$\begin{aligned} M_{\alpha\beta}(x_1,x_2,t) &= \int_{-b}^{b} x_3\sigma_{\alpha\beta}(x_1,x_2,x_3,t)dx_3, \\ & \qquad\qquad \alpha,\beta = 1,2. \qquad (3.25) \\ Q_\beta(x_1,x_2,t) &= \int_{-b}^{b} \sigma_{3\beta}(x_1,x_2,x_3,t)dx_3, \end{aligned}$$

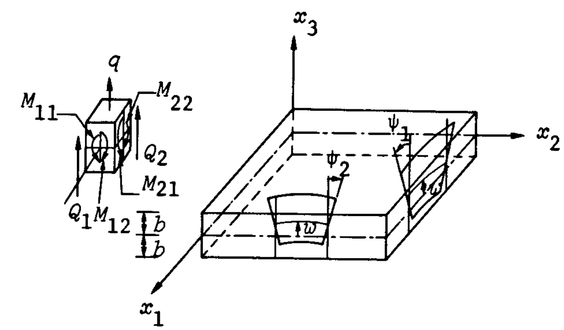

Fig. 3.3. Bending of a Plate

Assuming further that

$$M_{33}(x_1,x_2,t) = \int_{-b}^{b} x_3\sigma_{33}(x_1,x_2,x_3,t)dx_3 = 0, \qquad (3.26)$$

we obtain, upon multiplying the stress displacement equations (2.29) by x_3 and integrating over the thickness,

$$\begin{aligned} \sigma_{11}&: M_{11} = D\left(\frac{\partial\psi_1}{\partial x_1} + \nu\frac{\partial\psi_2}{\partial x_2}\right), \\ \sigma_{22}&: M_{22} = D\left(\frac{\partial\psi_2}{\partial x_2} + \nu\frac{\partial\psi_1}{\partial x_1}\right), \qquad (3.27a) \\ \sigma_{12}&: M_{12} = \mu I\left(\frac{\partial\psi_1}{\partial x_2} + \frac{\partial\psi_2}{\partial x_1}\right), \end{aligned}$$

where

$$I = 2b^3/3, \qquad D = 2\mu I/(1 - \nu) = 2Eb^3/3(1 - \mu^2),$$

and use has been made of (3.24) and (3.26) to express $\int x_3\varepsilon_{33}dx_3$ in terms of $\int x_3\varepsilon_{11}dx_3$ and $\int x_3\varepsilon_{22}dx_3$. Integrating the remaining two equations from $-b$ to b, we obtain

$$\sigma_{13}:\ Q_1 = 2\mu b\left(\psi_1 + \frac{\partial w}{\partial x_1}\right),$$
$$\sigma_{23}:\ Q_2 = 2\mu b\left(\psi_2 + \frac{\partial w}{\partial x_2}\right). \qquad (3.27b)$$

Equations (3.27), which relate the bending moment $M_{\alpha\beta}$ and shear force Q_α to the rotation of a cross section ψ_α and the deflection of the plate w, are a set of two-dimensional equations. They replace the general stress displacement relations.

The stress equation of motion can be modified accordingly. Multiplying the first two ($j = 1,2$) or (2.12) by x_3 and then integrating over the plate thickness for all three equations, we obtain

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{21}}{\partial x_2} - Q_1 = \rho I\,\frac{\partial^2\psi_1}{\partial t^2},$$
$$\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - Q_2 = \rho I\,\frac{\partial^2\psi_2}{\partial t^2}, \qquad (3.28)$$
$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} + q = 2\rho b\,\frac{\partial^2 w}{\partial t^2}.$$

The quantity $q = [\tau_{33}]_{-b}^{b}$ is the result of normal stresses applied to the upper and lower surfaces of the plate.

In the classical theory of the bending of plates, the effect of shear force on deflections is neglected, or equivalently the plate rigidity in resisting shear force is assumed infinite. This reduces (3.27b), with $Q_\alpha/2b\mu \to 0$, to

$$\psi_\alpha = -\frac{\partial w}{\partial x_\alpha}, \qquad \alpha = 1,2.$$

Physically, this means the angle of rotation of a cross section equals the slope of the mid-plane. Through this assumption, the moments are related directly to the curvature of the deflected plate by

$$M_{11} = -D\left(\frac{\partial^2 w}{\partial x_1^2} + \nu\,\frac{\partial^2 w}{\partial x_2^2}\right),$$
$$M_{22} = -D\left(\frac{\partial^2 w}{\partial x_2^2} + \nu\,\frac{\partial^2 w}{\partial x_1^2}\right), \qquad (3.29)$$
$$M_{12} = -2\mu I\,\frac{\partial^2 w}{\partial x_1\partial x_2} = -(1 - \nu)\,D\,\frac{\partial^2 w}{\partial x_1\partial x_2}.$$

Furthermore, in the classical theory, the effect of rotatory inertia, the term $\rho I\ddot{\psi}_\alpha$ in (3.28), is neglected. Thus the first two equations of (3.28) give rise to the magnitudes of shear force due to bending,

$$Q_1 = \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{21}}{\partial x_2} = -D\,\frac{\partial}{\partial x_1}\,\nabla^2 w,$$
$$Q_2 = \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} = -D\,\frac{\partial}{\partial x_2}\,\nabla^2 w, \qquad (3.30)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Substitution of the above into the third of (3.28) results in

$$D\nabla^2\nabla^2 w + 2b\rho \frac{\partial^2 w}{\partial t^2} = q. \tag{3.31}$$

This is the classical equation of plates in bending.

If the motion is harmonic in time with frequency ω, the deflection has the form

$$w(x_1,x_2,t) = W(x_1,x_2)e^{-i\omega t}$$

and W satisfies the equation ($q = 0$)

$$(\nabla^2\nabla^2 - \gamma^4)W = 0, \qquad \gamma^4 = \frac{2b\rho}{D}\omega^2. \tag{3.32}$$

Let $W = W_1 + W_2$. The above equation is satisfied if

$$\begin{aligned}(\nabla^2 + \gamma^2)W_1 &= 0,\\ (\nabla^2 - \gamma^2)W_2 &= 0.\end{aligned} \tag{3.33}$$

The first is a Helmholtz equation, and $W_1e^{-i\omega t}$ represents the wave motion in the plate. The W_2 represents a wave attenuating as it progresses. Combination of these two parts constitutes the *flexural wave* in plate.

The phase velocity of plane harmonic flexural waves is

$$c_f = \omega/\gamma = \gamma(D/2b\rho)^{\frac{1}{2}} = \gamma b c_s[2/3(1-\nu)]^{\frac{1}{2}}. \tag{3.34}$$

The wave is dispersive, as the wave velocity depends on the wavelength $(2\pi/\gamma)$. As in the case of generalized plane stress approximation, the classical theory for the bending of plate is also limited to low frequencies ($<0.3c_s/b$) and long wavelength ($>\pi b$).[3.4]

Derived above are the basic equations of motion for the different approximations. These equations will be used to solve specific problems for various geometrics and boundary conditions, and for steady-state, as well as for transient responses.

In the following section, a method for determining steady-state response and transient response will be presented.

4. STEADY-STATE AND TRANSIENT RESPONSES OF AN ELASTIC SOLID

IN THE PRECEDING SECTIONS we derived the displacement equations of motion for a homogeneous, isotropic linear elastic solid and the various two-dimensional approximations. Symbolically, the equations (2.29), (3.3), (3.10), (3.22), and (3.31) can be represented as

$$L\{u(x_i,t)\} + \rho f_b(x_i,t) = \rho\ddot{u}(x_i,t), \tag{4.1a}$$

where u is a scalar or a vector, ρf_b is the body force, and L is a linear differential operator. When the body force is neglected, Eq. (4.1a) becomes

$$L\{u(x_i,t)\} = \rho\ddot{u}(x_i,t). \tag{4.1b}$$

It might be of interest to note here that we have not mentioned any external force other than the body force. Of course the motion of the elastic body will depend on the nature of the external force, $f(x_i,t)$. The applied force may arise as an internal or surface source, or in the form of boundary conditions and initial conditions. Thus we may consider the problem from two different points of view. If the external forces are treated as sources, then the governing equation becomes

$$L\{u(x_i,t)\} = \rho\ddot{u} - f(x_i,t), \tag{4.1c}$$

where $f(x_i,t)$ describes the source density, giving not only the distribution of sources in space but also the time dependence at each point in space. Since we have considered all applied forces as sources, Eq. (4.1c) must be accompanied by a homogeneous boundary condition.

Contrary to the above, let us now consider the applied force $f(x_i,t)$ as boundary and initial conditions. Then, the governing equation is the same as Eq. (4.1b)--in other words, a homogeneous equation. However, the boundary conditions which are associated with the problem now become inhomogeneous.

A fundamental property for the linear differential Eq. (4.1) is that, if u' and u'' are solutions, i.e., $L\{u'\} = \rho\ddot{u}'$ and $L\{u''\} = \rho\ddot{u}''$, then $u = c_1u' + c_2u''$ is also a solution, where c_1 and c_2 are arbitrary constants. In general, if

$$L\{u^{(n)}\} = \rho\ddot{u}^{(n)} - f, \qquad n = 1,2,3,\ldots,$$

then

$$u = \sum_n c_n u^{(n)} \tag{4.2}$$

also satisfies Eq. (4.1). We shall on occasion replace the sum by an integral.

We have seen through the formulation above that the motion $u(x_i,t)$ is caused by an externally applied force $f(x_i,t)$, viewed either as sources or as boundary conditions. In both cases, we shall call all the applied forces *source* or *input*, and the consequent motion of the elastic body *response* or *output*. Determination of the *response* for a given *input* is our primary concern.

The input-output relationship for an elastic body is complicated, because u and f are functions of spatial coordinates x_i, as well as functions of time t. However, in our present discussion on the steady-state responses or transient responses, we need not include such complications. We shall focus our attention on the response of a fixed spatial point, and examine its time history as a result of the applied sources.

With regard to time dependence of the response, an elastic body is assumed to follow two basic conditions: the *condition of causality* and the *condition of time invariance*. A function $f(t)$ is *causal* if

$$f(t) = \begin{cases} 0, & t < 0, \\ f(t), & t > 0. \end{cases} \tag{4.3}$$

The condition of causality states that if the input $f(t)$ is causal, the output $u(t)$ is also causal, i.e.,

$$u(t) = \begin{cases} 0, & t < 0, \\ u(t), & t > 0. \end{cases}$$

As is clear from this condition, the problem of an elastic body with time dependent initial stresses is ruled out.

The condition of time invariance is that if $u(t)$ is the response for the source $f(t)$, then the response to $f(t - t_1)$ is $u(t - t_1)$ where t_1 is any constant. This condition is satisfied if the coefficients of the linear operator L are time independent, as is true for an elastic body with constant material constants.

As a function of time, the response will be classified as either *steady-state* or *transient*, as mentioned briefly in Section 2.6. We shall elaborate the properties and relationships of these responses in the subsequent discussion. We have left out much of the mathematical rigor in the analysis of the *transient response*, such as the integrability of a function and the approach to singular functions from the theory of distributions. We shall present only the techniques which will be used repeatedly in the rest of the book.

4.1. *Steady-State Response*

Let us consider first the steady-state response, since it is mathematically less complex than the transient response. By *steady state* we mean a response that is simple harmonic in an unbounded time domain. Usually, it is represented by

$$u(t) = Ae^{\pm i\omega t}, \qquad -\infty < t < \infty, \tag{4.4}$$

where A is a complex number. As mentioned in subsection 2.6, only the real or the imaginary part of (4.4) is taken to represent the motion.

Simple harmonic motion of an elastic solid can arise either during

free or during forced vibrations. For a bounded medium, it is possible to start a system by giving it an initial displacement or an initial velocity (impulse) which is compatible with one of the principal modes of the body. The subsequent free vibration is simple harmonic with a frequency equal to the corresponding principal frequency. Since the principal modes are usually very complicated for an elastic solid, excitation of this type seldom occurs. On the other hand, a simple harmonic source or simple harmonic forces applied at boundaries will generate forced motion of the system. After sufficient time has elapsed, the irregular initial disturbance of the forced motion dies out owing to a small damping inherent in all systems. What remains is a simple harmonic motion of the same frequency as the source.

In either case, the dependence on time for the response may be separated off as

$$u(x_i,t) = U(x_i,\omega)e^{\pm i\omega t}, \tag{4.5}$$

where $U(x_i,\omega)$ is a function of the spatial coordinates for a given frequency. The problem of determining the steady-state response for a source

$$f(x_i,t) = F(x_i,\omega)e^{\pm i\omega t}, \tag{4.6}$$

or the corresponding excitation appearing in the boundary condition, is then reduced to solving the equation

$$L\{U(x_i,\omega)\} = -\rho\omega^2 U(x_i,\omega) \tag{4.7}$$

with the appropriate boundary conditions.

When the displacement potentials φ and Ψ are used as in (2.31), the wave equations reduce to the familiar Helmholtz equation for steady-state response. We note that no initial conditions arise here due to the steady-state motion. We have chosen the zero time of observation long after the initiation of the motion. As far as the observer is concerned, the harmonic motion was started at time $t = -\infty$.

When the source has a magnitude of unity, i.e., $f(t) = e^{\pm i\omega t}$, the coefficient of $e^{\pm i\omega t}$ in the steady-state response is called the *admittance* of the elastic system. In terms of Eqs. (4.5) through (4.7), we may define the *admittance* as

$$\chi(x_i,\omega) = U(x_i,\omega)/F(x_i,\omega). \tag{4.8}$$

Knowing the admittance χ of an elastic body and the magnitude of the external source F, the steady-state response is simply

$$u(x_i,t) = F(x_i,\omega)\chi(x_i,\omega)e^{\pm i\omega t}.$$

If we are interested only in the steady-state response to simple harmonic forces there would be nothing more to say, and we would go directly to the specific problems. However, in many problems of practical interest, we are more interested in the response of an elastic system to an aperiodic disturbance, or in cases when the source is applied to the body suddenly. Therefore, we shall now address ourselves to the *transient response* problem.

The Fourier transform, and the related Laplace transform, are techniques used frequently for solving the transient problem. In the following subsections we shall discuss both techniques.

4.2. *Fourier Transform*

The Fourier transform $F(\omega)$ is defined by

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt. \tag{4.9a}$$

(See Refs. (2.3), (4.1), and (4.2) for a class of $f(t)$ such that $F(\omega)$ exist.) The function $f(t)$ can be determined from $F(\omega)$ according to the Fourier integral theorem:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} e^{i\omega(\tau-t)}f(\tau)d\tau. \tag{4.10}$$

Introducing $F(\omega)$, the above may be written as

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega. \tag{4.9b}$$

Equation (4.9b) is often referred to as Fourier's inversion formula, and $f(t)$ is called the inverse Fourier transform of $F(\omega)$.

The pair of equations (4.9) then represents the fundamental set of relations between the time-dependent function $f(t)$ and the frequency-dependent function $F(\omega)$. In general, the function of $F(\omega)$ is complex:

$$F(\omega) = R(\omega) + iX(\omega) = A(\omega)e^{i\varphi(\omega)}.$$

where $A(\omega)$ is called the Fourier spectrum of $f(t)$ and $\varphi(\omega)$ is its phase angle.

In what follows, we shall list some of the useful simple theorems and formulas for the Fourier transform.

(a) *Linearity*. If $F_1(\omega)$ and $F_2(\omega)$ are the Fourier transform of $f_1(t)$ and $f_2(t)$, respectively, and a_1 and a_2 are arbitrary constants, then

$$a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega). \qquad (4.11)$$

The notation $\leftrightarrow$ is used here to indicate that function $f(t)$ and $F(\omega)$ are related by Eq. (4.9).

(b) *Time Scaling*. If a is a real constant, then

$$f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right). \qquad (4.12)$$

(c) *Time and Frequency Shifting*. If the function $f(t)$ is shifted by a constant t_o, then its Fourier spectrum remains the same, but a term $t_o\omega$ is added to the phase angle

$$f(t - t_o) \leftrightarrow F(\omega)e^{+it_o\omega} = A(\omega)e^{i[\varphi(\omega)+t_o\omega]}, \qquad (4.13)$$

and if the $F(\omega)$ is shifted by a real constant ω_o, then

$$e^{-i\omega_o t} f(t) \leftrightarrow F(\omega - \omega_o). \qquad (4.14)$$

(d) *Time and Frequency Differentiation*. The Fourier transform of the n^{th} derivative of $f(t)$ is

$$\frac{d^n f(t)}{dt^n} \leftrightarrow (-i\omega)^n F(\omega). \qquad (4.15)$$

Here $f(t)$ and all its derivatives up to $(n-1)$th order are assumed to vanish as $|t| \to \infty$. This class of functions is particularly interesting since in almost all physical problems the disturbance

dies out in time. Likewise, differentiating $F(\omega)$ with respect to ω gives

$$(it)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n}. \qquad (4.16)$$

(e) *Moment Theorem*. This theorem relates the derivatives of $F(\omega)$ at $\omega = 0$ to the moments of its inverse transform. The n^{th} moment m_n of $f(t)$ is defined by

$$m_n = \int_{-\infty}^{\infty} t^n f(t)dt, \qquad n = 0,1,2,3,\ldots, \qquad (4.17)$$

and the theorem states that

$$(i)^n m_n = \left.\frac{d^n F(\omega)}{d\omega^n}\right|_{\omega=0}, \qquad n = 0,1,2,3,\ldots. \qquad (4.18)$$

It follows that the 0^{th} moment is the area under the curve of $f(t)$ and represents the value of $F(\omega)$ at $\omega = 0$. Thus, the slope $df/d\omega$ at $\omega = 0$ is represented by the first moment of $f(t)$, etc.

Similarly we may define the n^{th} moment M_n of $F(\omega)$ as

$$M_n = \int_{-\infty}^{\infty} \omega^n F(\omega)d\omega. \qquad (4.19)$$

With Eq. (4.19), the derivatives of $f(t)$ at the origin, $t = 0$, are related to the moments of the Fourier transform $F(\omega)$ by

$$(-i)^n M_n = \left.\frac{d^n f}{dt^n}\right|_{t=0}, \qquad n = 0,1,2,3,\ldots. \qquad (4.20)$$

Analogous to the previous discussion of m_n, the value of $f(t)$ at $t = 0$ is then determined by the area under the curve $F(\omega)$, and the slope df/dt, at $t = 0$, is determined by the first moment. That is

$$\left.\frac{df}{dt}\right|_{t=0} = M_1 = \int_{-\infty}^{\infty} \omega F(\omega)d\omega. \tag{4.21}$$

Equations (4.18) and (4.20) furnish powerful tools for evaluating the behavior either of $F(\omega)$ or $f(t)$ at $\omega = 0$ or $t = 0$, if one of the functions is completely defined. Furthermore, they are used sometimes as criteria in the approximate techniques. (See Chapter IV.)

(f). *The Time and Frequency Convolution Theorem* (Faltung). Last is the convolution theorem. Let $F_1(\omega)$ and $F_2(\omega)$ be the transforms of $f_1(t)$ and $f_2(t)$, respectively. Then the inverse transform of the product $F_1(\omega) \cdot F_2(\omega)$ is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}F_1(\omega)F_2(\omega)d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(\tau)f_2(t - \tau)d\tau$$

$$= f_1(t)*f_2(t). \tag{4.22}$$

Alternatively, the results can be written as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}F_1(\omega)F_2(\omega)d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(\tau)f_1(t - \tau)d\tau$$

$$\equiv f_2(t)*f_1(t). \tag{4.23}$$

An integral of this form is called a convolution integral of f_1 and f_2 (or a Faltung of f_1 and f_2).

In a similar manner, the Fourier transform $F(\omega)$ of the product of two functions $f_1(t)f_2(t)$ is equal to the convolution $F_1(\omega)*F_2(\omega)$ of their respective transforms $F_1(\omega)$ and $F_2(\omega)$:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(t)f_2(t)e^{i\omega t}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_1(\lambda)F_2(\omega - \lambda)d\lambda$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_2(\lambda)F_1(\omega - \lambda)d\lambda. \tag{4.24}$$

For convenience in future reference, the equations discussed above are presented in Table 4.1.

Table 4.1

PROPERTIES OF FOURIER TRANSFORM

$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$	$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$
$a_1f_1(t) + a_2f_2(t)$	$a_1F_1(\omega) + a_2F_2(\omega)$
$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
$f(t - t_o)$	$F(\omega)e^{i\omega t_o}$
$e^{-i\omega_o t} f(t)$	$F(\omega - \omega_o)$
$d^nf(t)/dt^n$	$(-i\omega)^nF(\omega)$
$(it)^nf(t)$	$d^nF(\omega)/d\omega^n$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}d\omega \qquad \frac{1}{\sqrt{2\pi}}$$

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \qquad -\frac{1}{\sqrt{2\pi}}\frac{1}{i\omega}$$

Convolution Theorem:

$$f_1(t)*f_2(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(t)f_2(t-\tau)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(t-\tau)f_2(t)d\tau$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}F_1(\omega)F_2(\omega)d\omega$$

$$F_1(\omega)*F_2(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_1(\lambda)F_2(\omega-\lambda)d\lambda = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_1(\omega-\lambda)F_2(\lambda)d\lambda$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(t)f_2(t)e^{i\omega t}dt$$

Moment Theorem:

Define

$$m_n = \int_{-\infty}^{\infty} t^n f(t)dt \quad \text{and} \quad M_n = \int_{-\infty}^{\infty} \omega^n F(\omega)d\omega$$

Then

$$i^n m_n = \left.\frac{d^n F(\omega)}{d\omega^n}\right|_{\omega=0}, \qquad (-i)^n M_n = \left.\frac{d^n f}{dt^n}\right|_{t=0}, \qquad n = 0,1,2,\ldots$$

4.3. *Special Functions and Their Fourier Transforms*

So far we have listed only basic formulas of the Fourier transform; no mention has been made of the type of function $f(t)$ that will be of most interest. In dealing with transient responses of elastic bodies which follow the condition of causality, the causal function is of primary interest.

First, we note that a function can always be decomposed into a sum of an even and odd part:

$$f(t) = \tfrac{1}{2}[f(t) + f(-t)] + \tfrac{1}{2}[f(t) - f(-t)] = f_e(t) + f_o(t).$$

If a Fourier transform is decomposed into a real and an imaginary part,

$$F(\omega) = R(\omega) + iX(\omega), \tag{4.25}$$

$$R(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\omega t dt,$$

$$X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\omega t dt.$$

An even function $f_e(t)$ satisfying the condition

$$f_e(t) = f_e(-t),$$

has the transform $X(\omega) = 0$ and

$$R(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_e(t)\cos\omega t dt. \tag{4.26a}$$

The inverse transform is

$$f_e(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} R(\omega)\cos\omega t d\omega. \tag{4.26b}$$

Similarly, an odd function $f_o(t) = -f_o(-t)$ has the transform pair

$$X(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f_o(t)\sin\omega t\,dt, \tag{4.27a}$$

$$f_o(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty X(\omega)\sin\omega t\,d\omega, \tag{4.27b}$$

and $R(\omega) = 0$.

A causal function $f(t)$ (Eq. 4.3) may be treated either as an even function $f_e(t)$ or as an odd function $f_o(t)$ over the range $-\infty < t < \infty$ (Fig. 4.1). For the time interval of interest, $t > 0$,

$$f(t) = 2f_e(t) = 2f_o(t), \qquad t > 0, \tag{4.28}$$

and the transform of the causal function can be calculated from either (4.26a) or (4.27a), with $f_e(t)$ or $f_o(t)$ replaced by $(\frac{1}{2})f(t)$.

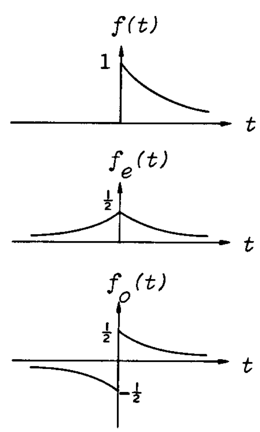

Fig. 4.1. Even and Odd Parts of $f(t)$

Furthermore, as can be seen from (4.28), (4.26b), and (4.27b),

$$f(t) = 2\sqrt{\frac{2}{\pi}}\int_0^\infty R(\omega)\cos\omega t\,d\omega = 2\sqrt{\frac{2}{\pi}}\int_0^\infty X(\omega)\sin\omega t\,d\omega, \qquad t > 0. \tag{4.29}$$

Thus the real and imaginary parts of the Fourier transform of a causal function are not independent of each other, but one of them can be computed from the other.

Next, we shall present some special properties of two singular functions. These functions are of significance because once the response of a system to these functions is known, the responses of the system to arbitrary inputs are determinable, according to the convolution theorem, by a simple integration.

(a) *Dirac Delta Function* $\delta(t)$:

One may describe $\delta(t)$ loosely as a function which is zero everywhere except at $t = 0$, where it is infinite, and which has a unit area under its graph.

$$\left.\begin{aligned} \delta(t) &= 0, \qquad |t| > 0, \\ \int_{-\infty}^{\infty} \delta(t)dt &= 1. \end{aligned}\right\} \tag{4.30}$$

We shall define the delta function $\delta(t)$ by the following integral for an arbitrary function $\varphi(t)$:

$$\int_{-\infty}^{\infty} \delta(t-\tau)\varphi(t)dt = \varphi(\tau); \tag{4.31}$$

$\varphi(t)$ is assumed to be continuous at $t = \tau$. Similarly, the n^{th} derivative of $\delta(t)$ is defined by

$$\int_{-\infty}^{\infty} \frac{d^n\delta(t-\tau)}{dt^n}\varphi(t)dt = (-1)^n \left.\frac{d^n\varphi(t)}{dt^n}\right|_{t=\tau}. \tag{4.32}$$

The Fourier transform of $\delta(t)$ is:

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}},$$
$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}d\omega. \tag{4.33}$$

(b) *Heaviside Unit Step Function* $h(t)$:

A function closely related to the delta function is the Heaviside unit function $h(t)$, defined by

$$h(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0, \end{cases} \tag{4.34}$$

and

$$h(t) = \int_{-\infty}^{\infty} \delta(\tau)d\tau, \qquad h'(t) = \delta(t). \tag{4.35}$$

The Fourier transform of $h(t)$ is

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(t)e^{i\omega t}dt = -\frac{1}{\sqrt{2\pi}}\frac{1}{i\omega}; \tag{4.36}$$

$$h(t) \leftrightarrow -\frac{1}{\sqrt{2\pi}}\frac{1}{i\omega}. \tag{4.37}$$

4.4. *Transient Response*

At the end of subsection 4.1 we showed that for a *linear* elastic system, the steady-state response to a simple harmonic force $F(\omega)e^{-i\omega t}$ is $F(\omega)\chi(x_i,\omega)e^{-i\omega t}$, where $\chi(x_i,\omega)$ is the admittance of the system. Now, to find the transient response of an aperiodic disturbance, $f(t)$, we first break up $f(t)$ into its simple harmonic components by means of the Fourier integral

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega, \qquad F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt.$$

Then, having solved the steady-state problem to obtain the admittance, $\chi(x_i,\omega)$, we superimpose the components to obtain the response of the system resulting from the original aperiodic force $f(t)$:

$$u(x_i,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \chi(x_i,\omega)F(\omega)e^{-i\omega t}d\omega. \tag{4.38}$$

If the input is a unit impulse, recall that

$$\delta(t) \leftrightarrow \frac{1}{\sqrt{2\pi}}.$$

Then the systems response to $\delta(t)$ is simply

$$u_\delta(x_i,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(x_i,\omega)e^{-i\omega t}d\omega. \tag{4.39}$$

Due to a delta function input, we refer to $u_\delta(x_i,t)$ as the *impulse response*. It should be apparent now that the admittance function $\chi(x_i,\omega)/\sqrt{2\pi}$ and the impulse response $u_\delta(x_i,t)$ form a Fourier transform pair.

For an arbitrary input, we may determine the system response $u(x_i,t)$ by Eq. (4.38), or if we wish, use the convolution theorem as given by Eq. (4.23). Thus the response to arbitrary input $f(t)$ is:

$$u(x_i,t) = \int_{-\infty}^{\infty} f(\tau)u_\delta(x_i,t-\tau)d\tau, \tag{4.40}$$

or

$$u(x_i,t) = \int_{-\infty}^{\infty} u_\delta(x_i,\tau)f(t-\tau)d\tau. \tag{4.41}$$

Since the delta function is causal, from the condition of causality the impulse response is also causal. Thus

$$u_\delta(x_i,t-\tau) = 0 \qquad \text{when } t < \tau.$$

It then follows that Eqs. (4.40) and (4.41) can be rewritten as

$$u(x_i,t) = \int_{-\infty}^{t} f(\tau)u_\delta(x_i,t-\tau)d\tau, \tag{4.42}$$

and

$$u(x_i,t) = \int_{0}^{\infty} f(t-\tau)u_\delta(x_i,\tau)d\tau. \tag{4.43}$$

If, in addition, $f(\tau)$ is causal, with $f(t-\tau) = 0$ when $t < \tau$, then the above equations are reduced to the familiar *Duhamel integral*:

$$u(x_i,t) = \int_{0}^{t} f(\tau)u_\delta(x_i,t-\tau)d\tau, \tag{4.44}$$

or

$$u(x_i,t) = \int_{0}^{t} u_\delta(x_i,\tau)f(t-\tau)d\tau. \tag{4.45}$$

Equations (4.44) and (4.45) are a statement of the principle of superposition in time as illustrated in Fig. 4.2.

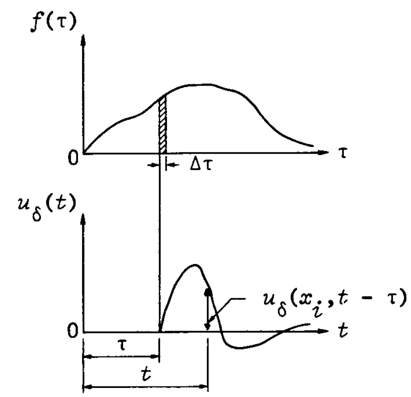

Fig. 4.2. Illustration of the Duhamel Integral and Unit Impulse Response

Using the formulas given above we may now derive the other familiar form of Duhamel's integral, that of the response due to arbitrary input in terms of the response to a Heaviside unit function, $h(t)$, Eq. (4.34). Letting $u_h(x_i,t)$ denote the *indicial response*, that is, the response to a unit step function $h(t)$, then according to (4.45) we have

$$u_h(x_i,t) = \int_{0}^{t} u_\delta(x_i,\tau)d\tau, \tag{4.46}$$

or

$$u_h'(x_i,t) = \frac{du_h(x_i,t)}{dt} = u_\delta(x_i,t). \tag{4.47}$$

It follows from (4.44) that the response to an arbitrary $f(t)$ can be written as

$$u(x_i,t) = \int_0^t f(\tau)u_h'(t-\tau)d\tau, \tag{4.48}$$

or after integrating by parts once, we have

$$u(x_i,t) = f(0)u_h(t) + \int_0^t f'(\tau)u_h(t-\tau)d\tau. \tag{4.49}$$

Equation (4.49) is another form of Duhamel integral. Again it is merely a statement of the method of superposition in time as illustrated by Fig. 4.3.

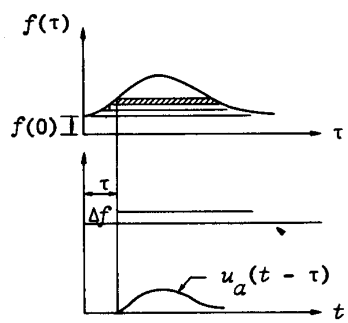

Fig. 4.3. Illustration of Duhamel's Integral and Step Response

To summarize, the transient response $u(t)$ of an elastic body due to an arbitrary source $f(t)$ can be determined by one of the following three ways:

1. Finding the steady-state response or the admittance $\chi(\omega)$ in (4.38) of the same body and the Fourier spectrum $F(\omega)$ of $f(t)$, (4.9), then using (4.38):

$$u(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\chi(\omega)e^{-i\omega t}d\omega; \tag{4.50}$$

2. Finding the impulse response $u_\delta(t)$ due to a delta function source, then using (4.43):

$$u(t) = \int_0^t f(t-\tau)u_\delta(\tau)d\tau; \tag{4.51}$$

The functions $u_\delta(t)$ and $\chi(\omega)/\sqrt{2\pi}$ form a Fourier transform pair (4.39)

$$u_\delta(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\chi(\omega)e^{-i\omega t}d\omega, \tag{4.52a}$$

$$\frac{1}{\sqrt{2\pi}}\chi(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_\delta(t)e^{i\omega t}dt; \tag{4.52b}$$

3. Finding the indicial response $u_h(t)$ due to a unit step function source, then using (4.49) for the causal function $f(t)$:

$$u(t) = f(0)u_h(t) + \int_0^t f'(\tau)u_h(t-\tau)d\tau. \tag{4.53}$$

Indicial response and impulse response are related by (4.46) and (4.47)

4.5. *Laplace Transform*

In dealing with transient problems, the source functions in most cases are causal. By the condition of time invariance, the response is also a causal function. For problems of this type, the Laplace transform is frequently used. The Laplace transform $F_\ell(s)$ of a function $f(t)$ is defined as

$$F_\ell(s) = \int_0^\infty f(t)e^{-st}dt. \tag{4.54}$$

The inversion formula is

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F_\ell(s)e^{st}ds, \tag{4.55}$$

where γ is a real constant which can have any value such that the path of integration on the complex s-plane lies to the right of all singularities of $F_\ell(s)$.

The above pair of transforms can be derived formally from the Fourier integral. Consider the function

$$f_1(t) = \begin{cases} e^{-\gamma t}f(t), & \gamma > 0, \quad t > 0, \\ 0, & t < 0. \end{cases} \tag{4.56}$$

Replacing $f_1(t)$ in the Fourier integral formula (4.10)

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} e^{i\omega(\tau - t)}f_1(\tau)d\tau,$$

by (4.56), we have

$$f(t) = \frac{1}{2\pi}\, e^{\gamma t}\int_{-\infty}^{\infty} e^{-i\omega t}d\omega \int_0^\infty f(\tau)e^{-(\gamma-i\omega)\tau}d\tau.$$

If in the above we set $s = \gamma - i\omega$ and

$$F_\ell(s) = \int_0^\infty f(\tau)e^{-s\tau}d\tau,$$

then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\ell(s)e^{(\gamma-i\omega)t}d\omega$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F_\ell(s)e^{st}ds.$$

The last equation is the inversion formula (4.55). It is thus seen that the Laplace transform is but another form of Fourier transform. In fact, for causal function $f(t)$, $F_\ell(s)$ in (4.54) is just $\sqrt{2\pi}$ times the Fourier transform $F(\omega)$ when s is identified with $-i\omega$.

Equations analogous to (4.11) through (4.24) can also be derived for the Laplace transform. They are listed in Table 4.2.

Table 4.2

PROPERTIES OF LAPLACE TRANSFORM

$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds$	$F(s) = \int_0^{\infty} f(t)e^{-st}dt$
$a_1f_1(t) + a_2f_2(t)$	$a_1F_1(s) + a_2F_2(s)$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
$f(t - t_o,$	$e^{-t_o s}F(s)$
$e^{s_o t}f(t)$	$F(s - s_o)$
$\frac{d^n f(t)}{dt^n}$	$s^nF(s) - s^{n-1}F(+0)\ldots$ $\ldots F^{(n-1)}(+0)$
$t^n f(t)$	$(-1)^n \frac{d^nF}{ds^n}$
$\delta(t)$	1
$h(t)$	$\frac{1}{s}$

Convolution Theorem:

$$f_1(t)*f_2(t) = \int_0^t f_1(t - \tau)f_2(\tau)d\tau \qquad F_1(s)F_2(s)$$

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